

# Dynamic Pricing via Dynamic Programming<sup>1</sup>

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**Abstract.** This article specifies an efficient numerical scheme for computing optimal dynamic prices in a setting where the demand in a given period depends on the price in that period, cumulative sales up to the current period, and remaining market potential. The problem is studied in a deterministic and monopolistic context with a general form of the demand function. While traditional approaches produce closed-form equations that are difficult to solve due to the boundary conditions, we specify a computationally tractable numerical procedure by converting the problem to an initial-value problem based on a dynamic programming formulation. We find also that the optimal price dynamics preserves certain properties over the planning horizon: the unit revenue is linearly proportional to the demand elasticity of price; the unit revenue is constant over time when the demand elasticity is constant; and the sales rate is constant over time when the demand elasticity is linear in the price.

**Key Words.** Dynamic pricing, dynamic programming, optimal control.

## 1. Introduction

Following Robinson and Lakhani's seminal work of four decades ago (Ref. 1), the determination of dynamic pricing for new product planning has been a central theme in market research. By modeling the effect of the current price and sales on future sales and production, dynamic pricing studies seek the optimal pricing strategy over time to maximize the total

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profit over the planning horizon. The problem becomes more challenging with considerations of oligopolistic competition (Refs. 2, 3), the impact of market penetration rate and the speed of diffusion on the pricing strategies (Ref. 4), effects of advertising and demand uncertainty (Refs. 5, 6). The readers are referred to Refs. 7–9 for thorough reviews of the related literature.

This article considers an optimal control perspective on this problem, where production, sales, and product adoption are controlled via prices in each period. Our work is set in the context of optimal dynamic pricing for a monopoly under deterministic demand (as in Ref. 1, but we adopt a more general form for the demand function). In most models, the optimal prices are determined as solutions to systems of simultaneous differential equations with boundary conditions, due to which the computation of the operational numerical solutions is generally quite challenging. For instance, a popular tool in the literature is the Pontryagin maximum principle, where the resulting Euler equations are difficult to solve numerically (Ref. 10). This article specifies an efficient numerical scheme obtained by converting the problem to an initial-value problem based on a dynamic programming formulation (Ref. 11). In addition, we solve the problem using the Pontryagin maximum principle to obtain insightful economic interpretations as well as validation for alternative numerical procedures. We show that the unit revenue from production (or the discounted unit revenue if discounting is considered) should be linearly proportional to the demand elasticity of price at any time  $t$  over the entire planning horizon. When the demand function exhibits constant elasticity toward price, the optimal pricing strategy results in constant unit revenue over time, while the optimal sales rate is constant over time when the demand elasticity is linear in the price.

## 2. Mathematical Model

Our basic notations follows Ref. 1:  $q(t)$  is the cumulative amount of product that has been sold up to time  $t$ ; this is the installed base at time  $t$ . The quantity  $p(t)$  is the price at time  $t$ ,  $c(q)$  is the unit production cost when the cumulative production is  $q$ , and  $f(p(t), q(t))$  is the rate of sales at time  $t$ , given the price  $p(t)$  and the size of the installed base  $q(t)$ .

The general demand function  $f(p, q)$ , also called adoption rate in studies of innovation diffusion, covers various factors that affect sales, including production diffusion, positive network externalities (higher demand when installed base is large), and congestion effects (lower demand when there are more users). The quantity  $q(t)$  describes the state

of the market and  $p(t)$  is the control variable that needs to be optimized. Our goal is to find the optimal pricing strategy that maximizes the total profit over a given period of time  $[0, T]$ . The formulation is

$$\pi^* = \max_{p(t)} \int_0^T (p - c)f(p, q)dt, \tag{1}$$

$$\text{s.t. } dq/dt = f(q, p), \tag{2a}$$

$$q(0) = q_0. \tag{2b}$$

The unknowns of problem (1)–(2) are often solved using the Pontryagin maximum principle. We form the Hamiltonian  $H$  as

$$H = (p - c)f(p, q) + \lambda f(p, q), \tag{3}$$

where  $\lambda$  is the adjoint variable (Lagrange multiplier) representing the future value shadow price for an additional adopter (Ref. 7). Let  $\dot{q}$  represent the total derivative of  $q$  with respect to  $t$ ; let  $H_x$  represent the partial derivative of function  $H$  with respect to the term  $x$ . Applying the maximum principle, the optimal values of the vectors  $q(t)$  and  $p(t)$  satisfy the simultaneous equations below,

$$\dot{q} = H_\lambda = f(p, q), \tag{4}$$

$$-\dot{\lambda} = H_q = (p - c)f_q + \lambda f_q, \tag{5}$$

$$H_p = f(p, q) + (p - c)f_p + \lambda f_p = 0, \tag{6}$$

with the boundary conditions

$$q(0) = q_0, \tag{7}$$

$$\lambda(T) = 0. \tag{8}$$

**2.1. Properties of the Optimal Schedule.** First, we use Eqs. (4)–(8) to derive the general relationship between the optimal production and pricing schedules. The total derivative of  $H$  with respect to  $t$  is

$$(d/dt)H(p, q, \lambda) = H_p \dot{p} + H_q \dot{q} + H_\lambda \dot{\lambda}.$$

Equations (4)–(6) yield

$$(d/dt)H(p, q, \lambda) = 0 + H_q H_\lambda + H_\lambda (-H_q) = 0.$$

Therefore,  $H$  remains constant over time. According to Eq. (6), we have

$$p - c + \lambda = -f/f_p. \quad (9)$$

Since the demand elasticity toward price is  $\varepsilon(p) = -p \cdot f_p/f$ , Eq. (9) yields the following interpretation of the optimal pricing schedule:

$$1/\varepsilon(p) = [p - (c - \lambda)]/p. \quad (10)$$

Compare this equation with the standard pricing rule

$$1/\varepsilon(p) = (p - c)/p$$

for a single period monopoly: the optimal price in each period is where the inverse elasticity equals the firm's pricing power (margin, divided by price). As recognized by Kalish (Ref. 7), the optimal dynamic pricing rule for the multiperiod monopolist is analogous to that for a single-period monopolist, except that the variable cost is adjusted by the shadow price  $\lambda$  to capture the impact on subsequent periods of over production or under-production in the current period.

Since  $H$  is constant over time, we can combine Eqs. (6) and (10) to get

$$f \cdot p = H(-pf_p)/f = H\varepsilon(p), \quad (11)$$

yielding that the revenue in each period  $f \cdot p$  is linearly proportional to the demand elasticity toward price. Once the constant  $H$  is identified at the beginning of the production process, the optimal pricing rule in Eq. (11) guides the optimal pricing and production schedules. Note that this rule holds at every time  $t$  for the entire planning horizon.

Incorporating time discounting into the model, with discount rate  $\beta$ , it is straightforward to derive the following counterpart of Eq. (11):

$$e^{-\beta t} f \cdot p = H(-pf_p)/f = H\varepsilon(p),$$

which means that the discounted unit revenue should be linearly proportional to the price elasticity.

**2.2. Special Cases.** Next, let us consider some special cases of the adoption rate  $f(p, q)$ .

**Linear Elasticity Function.** Assume that the adoption rate function is separable in  $p$  and  $q$ ,

$$f(p, q) = e^{-rp} Q(q), \quad (12)$$

where  $\varepsilon(p) = rp$  is linear in the price,  $r$  measures the price sensitivity, and  $Q$  is a general function of  $q$  that describes the effect of the installed base on the consumer purchase decision. This function generalizes the adoption rate function introduced by Ref. 1,

$$\dot{q} = (N - q)(a + bq)e^{-rp},$$

where  $N$  represents total market potential, which is independent of the price. Differentiating Eq. (12) with respect to  $p$  yields

$$f_p/f = -r,$$

which in light of Eq. (11) gives

$$f(p(t), q(t)) = rH, \tag{13}$$

meaning that the optimal sales rate is constant over time. If there is no storage between production and sales, then this also means that the factory should keep a constant production rate over the entire planning horizon. Note that the way of modeling the effect of market saturation level on adoption rate [i.e., the form of  $Q(q)$ ] does not change this observation.

**Constant Elasticity Function.** The constant elasticity function

$$f(p, q) = p^{-\alpha} Q(q)$$

was adopted by Bass (Ref. 12) and Bass and Bultez (Ref. 13) to describe the influence of the price on the adoption rate. In this case, Eq. (11) leads to

$$pf = -k\alpha,$$

suggesting that the unit revenue should remain constant if the optimal pricing strategy is followed.

**Other Functional Forms for Durable Goods.** Consider the adoption rate function

$$f(p, q) = \dot{q} = [N(p) - q]h(q),$$

where  $N(p)$  is the total number of individuals who are willing to buy at a certain price. The interpretation of this model is that the adoption rate is a product of the remaining market potential  $N(p) - q$  and the conditional likelihood of purchase  $h(q)$ . Substituting the demand function into Eq. (11), we have

$$f^2(p, q)/f_p = ((N(p) - q)^2 h^2(q))/N_p h(q) = [(N(p) - q)/N_p] f = -H.$$

If we assume that  $N(p)$  is linear in  $p$  (as in Ref. 8), then  $N_p$  is constant, yielding that the production rate is inversely proportional to the remaining market potential.

### 3. Computing the Numerical Solution via Dynamic Programming

As noted above, the optimal schedule is such that the inverse elasticity equals the pricing power condition, but the variable cost is adjusted by the shadow price  $\lambda$ , which itself is an unknown variable that needs to be solved simultaneously with the state and control variables in the Euler equations (4)–(8). These equations yield insightful properties of the optimal pricing strategy; however, we may encounter convergence and stability problems when employing them to compute the numerical prices and production schedule. Bellman and Kalaba (Ref. 10) recognized the difficulties in solving boundary-value problems and suggested alternative methods that are more suitable for digital computation. One approach is to convert the problem to an initial-value problem (Cauchy system) which involves only additions and multiplications and thus has computational stability as long as the problem is well-posed.

In order to obviate the computational problems associated with the Euler equations, we turn to the dynamic programming formulation (Bellman and Kalaba, Ref. 11). The problem is reframed as a multistage decision process for planning horizons of variable length. This approach leads to an initial-value problem for a system of ordinary differential equations. The resultant computational problem can be solved readily using any one of a number of excellent integration routines. This avoids the pitfalls of instability. A further advantage of dynamic programming is the easy handling of discrete problems which often arise in applications. For example, the production rate may take on discrete values and this reduces the computational complexity in the dynamic programming approach, whereas it increases the complexity under conventional variational approaches.

In applying the Bellman principle of optimality, we take the view that the planning period may be varied and that the problem is a multistage decision processes. We introduce the optimal return functional at any time  $t$  and any state  $q$ . Let us define the quantity  $\phi(t, q)$  as the maximum profit from the time  $t$  to the ending time  $T$ , given an amount  $q$  sold up to time  $t$ . Consider a very short time interval  $[t, t + dt]$ . The immediate return of setting the price as  $p$  at time  $t$  is  $(p - c)f(p, q)dt$ . The new state at time  $t + dt$  then becomes  $q + f(p, q)dt$ . The statement of the Bellman principle of optimality which sets forth the relations between the optimal return

functions is

$$\phi(t, q) = \max_p \{(p - c)f(q, p)dt + \phi(t + dt, q + f(q, p)dt)\}. \tag{14}$$

Using a Taylor series expansion yields

$$\phi(t, q) = \max_p \{(p - c)f(q, p)dt + \phi(t, q) + \phi_t dt + \phi_q f(q, p)dt + \text{HOT}\}. \tag{15}$$

Dividing both side of Eq. (15) by  $dt$  and letting  $dt \rightarrow 0$  yields

$$-\phi_t = \max_p \{X\}, \quad \text{where } X = (p - c)f(q, p) + \phi_q f(q, p).$$

To reach the extreme value of the term being maximized, we set

$$dX/dp = 0.$$

This yields the optimal control law

$$f(q, p) + (p - c + \phi_q)f_p(q, p) = 0. \tag{16}$$

Choosing prices optimally according to Eq. (8) yields the optimal return equation

$$-\phi_t = (p - c + \phi_q)f(p, q). \tag{17}$$

The initial condition of this dynamic programming formulation is  $\phi(T, q) = 0$ , obtained by setting  $t = T$ . The detailed steps of the numerical procedure for finding the optimal policy  $p(t)$  and the optimal return functions  $\phi(t, q)$  are specified below. The procedure involves the choice of the time interval  $dt$  and the quantity increment  $dq$ .

**3.1. Backward Sweep.** Let  $p(t, q)$  represent the optimal price at time  $t$  given the cumulative sales  $q(t)$ .

Step 0. Let  $t = T$ ;  $q = \{q_0, q_1, \dots, q_k, \dots, \bar{q}\}$ , where  $q_{k+1} = q_k + dq$  and  $\bar{q}$  is a predefined upper bound of  $q$ . Initialize  $\phi(T, q) = 0, \forall q$ .

Step 1. See Steps 1a–1d below.

Step 1a. Approximate  $\phi_q(t, q)$  for all  $q$ .

Step 1b. Calculate  $p(t, q)$  according to Eq. (16).

Step 1c. Calculate  $f(p, q)$  and  $f_p(p, q)$ .

Step 1d. Calculate  $\phi_t$  according to Eq. (17).

Step 2. Let  $t = t - dt$ . Update  $\phi(t, q) = \phi(t, q) - \phi_t dt$ .

Step 3. If  $t = 0$ , then stop; otherwise, go to Step 1.

**3.2. Forward Sweep.** The results from the backward sweep include the optimal value function  $\phi(t, q)$  and the optimal pricing strategy  $p(t)$  for each starting time  $t = \{0, dt, \dots, T\}$  with initial cumulative sales  $q = 0, dq, \dots, \bar{q}$ . The optimal prices and production amounts are determined sequentially via the following steps.

- Step 1. Let  $t = t_0$ , determine the optimal price  $p(t)$  in period  $t$  based on the cumulative sales up to period  $t$ ;
- Step 2. Calculate  $dq$  based on  $p(t)$  in period  $t$  and  $q(t+1) = q(t) + dq$ . The value of  $q(t+1)$  may lie in between  $[q_k, q_{k+1}]$  for some  $k$  and is determined through interpolation;
- Step 3. Let  $t = t + 1$ . Repeat Steps 1–2 until the final planning horizon is reached.

Given each  $t$  and  $q(t)$ , the unknown function  $\phi(t, q)$  is evaluated to compute the optimal profit for the duration  $[t, T]$ . In case the firm did not follow the optimal pricing strategy at the beginning of the sales, the solution from the backward sweep can still be used to determine the optimal pricing path for the remaining process starting from any given intermediate step.

The computational complexity and the accuracy of the procedure depend on the choice of the intervals  $dt$  and  $dq$ . Specifically, as the interval gets smaller, the complexity increases, while the objective function value approaches the true optimum. In principle, the production is asymptotically optimal, i.e.,

$$\lim_{dt \rightarrow 0, dq \rightarrow 0} \phi(t, q) = \pi^*,$$

where  $\pi^*$  is the optimal profit defined in Section 2. In practice, the choice of the interval  $dt$  is usually dictated by business consideration related to the cost of price change: market sensitivity toward frequent price changes, menu costs (costs of changing sticker prices), and other administrative effects of price instability. Despite recent advances in Internet and information technologies that automate the administration of price changes, these costs are quite significant. Therefore, business considerations force a larger interval for price changes than the interval size that poses a significant computational burden. In such cases, business considerations dominate computational concerns; hence, the intervals should be set to the smallest levels dictated by business considerations.

Raman and Chatterjee (Ref. 6) have shown that the deterministic formulation can be extended to cover a stochastic optimal pricing problem (with demand uncertainty) to maximize total expected gain over the planning period. This stochastic pricing problem can be solved using the same

procedure described above. However, in the stochastic case, no optimal pricing path can be predefined. The optimal pricing strategy should be obtained in a feedback control manner. At each decision step, the cumulative sales needs to be observed from the real market and the optimal price should be chosen according to the procedure defined above. The feedback control feature of dynamic programming allows accommodation of any real-time information acquired along the decision process; therefore it is suitable for stochastic situations.

#### 4. Numerical Results and Interpretation

We use the demand function

$$f(q, p) = e^{-0.1p}(0.2 + 2qe^{-2q})$$

to illustrate the firm pricing strategy. Here,

$$Q(q) = (0.2 + 2qe^{-2q});$$

if price were held constant, then  $Q(q)$  represents the effect of the installed base on the demand for the product. As depicted in Figure 1,  $Q(q)$  rises rapidly with  $q$  up to some value  $\hat{q}$  (indicating either word-of-mouth or positive network externality) and then falls (indicating market saturation or effect of congestion).

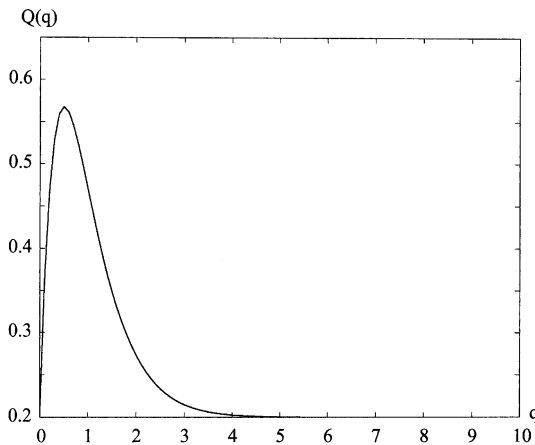


Fig. 1. Effect of the installed base on product demand for the demand function  $f(p, q)$ ;  $Q(q)$  represents this effect when price is held constant.

For our computations, the cost of producing one unit of product is set to be 4. The planning period is  $[0,10]$ . The optimal policy and optimal return function are evaluated at each  $t$  and  $q$  with an increment of 0.01. Note that the accuracy of the numerical procedure depends on the computational increment. Our results indicate that, with increment size 0.01, the sales rate is a constant  $(0.1058 \pm \xi, \text{ where } \xi = 0.0005)$  over the planning period, consistently with Eq. (13). If  $\xi$  were large relative to the sales rate, this would indicate an invalid output, suggesting that the model analyst should modify (reduce) the computational increment until  $\xi$  is sufficiently small.

**4.1. Optimal Price Path.** Figure 2 demonstrates the effect of the demand function on the firm pricing path over time. For both scenarios depicted in the figure ( $r = 0.1$  or  $r = 0.2$ ), we see that the firm must initially subsidize the product by offering a low price in order to stimulate demand; once there is a sufficiently large installed base, the firm can reduce the price subsidy. Finally, the firm needs to reduce price again when congestion effects dominate and reduce demand.

**4.2. Effect of Price Sensitivity on Optimal Price Path.** Figure 2 demonstrates also the effect of the price sensitivity parameter on the firm pricing strategy over time. Intuitively, greater price sensitivity reduces the firm ability to increase prices in the initial period. In the figure, we see that the rate of increase in the price is lower for  $r = 0.2$  than for  $r = 0.1$ . Conversely, once congestion effects dominate, the firm must make a greater reduction in price to induce demand when  $r$  is low. Thus, the lower price sensitivity of the market provides more flexibility for the firm to adjust its pricing scheme over the planning period. The same effect is observed for

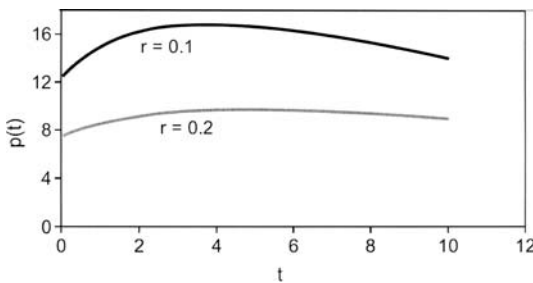


Fig. 2. Comparison between models with different levels of price sensitivity with adoption rate  $f(p, q)$ .

other demand functions, such as the constant elasticity function  $f(p, q) = p^{-\alpha} Q(q)$  in Section 2.2.

**4.3. Dynamic vs. Static Pricing.** To understand the impact and value of dynamic pricing, we compare the results of this approach with the simple case of static pricing, where the firm sets one price for the entire planning horizon. As it is well known, the standard optimal price  $p$  in static optimization problem satisfies

$$-(f/f_p)p = (p - c)/p,$$

which yields an optimal price  $p^* = 14$  for the static pricing policy. Figure 3 compares the prices, sales rate, and profit under the static and dynamic pricing policies. The dynamic pricing approach gives the manager multiple levers to control price and sales. Compared with static pricing, the firm sets a lower price in the initial period in order to stimulate demand and create a larger installed base [cumulative sales, Figure 3(b)]. In later periods  $[t \in [0.6, 1]$  in Figure 3(a)], the installed base allows the firm to

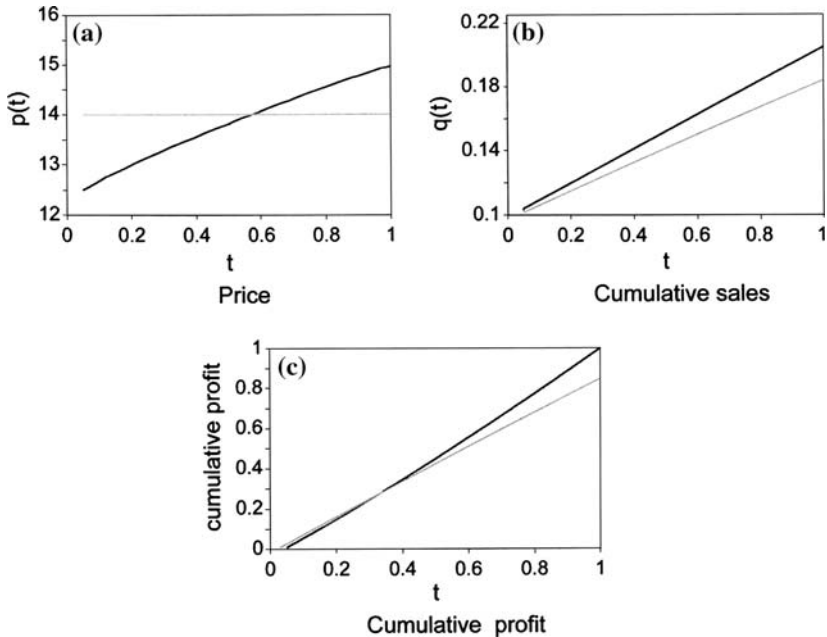


Fig. 3. Comparison of results from dynamic multiperiod and static single-period optimization.

maintain a greater sales rate  $dq/dt$  even though it charges a higher price than the constant static price.

Comparing the profits [Figure 3(c)], we see that the long-term planning implied under dynamic pricing allows the firm to achieve a higher cumulative profit. The firm strategically earns a lower profit in the initial period (due to the price subsidy), but the larger installed base improves the demand for the product and yields a higher profit in later periods. As the cumulative sales and price increases the cumulative profit gradually increases and soon exceeds the profit gained from the static model. However, the benefit from long-term planning is vulnerable to market uncertainties. If product demand does not increase as expected, then the firm may never recover the profit sacrificed in the early periods.

## 5. Conclusions

We have treated the dynamic pricing problem as an optimal control problem and shown that alternative methods can be used for solving the problem. Our analysis employed a context of single-product monopoly with deterministic demand, but the observations on optimal pricing and production patterns from this research are general and may be of value to other dynamic problems considering competition and demand uncertainty. While traditional approaches produce closed-form equations that are difficult to solve due to boundary conditions, we specify a computationally tractable numerical procedure by converting the problem to an initial-value problem based on a dynamic programming formulation. The choice of the solution methods depends on the type of equations that one prefers to solve ultimately and the type of results that one expects. Theoretically, the results from using classical variational methods and dynamic programming should not differ from each other under a deterministic situation. However, when uncertainty is involved, dynamic programming has the advantages of being capable of accommodating real-time information into decision making (feedback control perspective) and updating our knowledge about the uncertain environment along the decision-making processes (adaptive feedback control perspective). Adding learning aspects into stochastic dynamic pricing model will be a natural extension of this work.

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